COMPARING THE DISTRIBUTIONS, SPECIFICALLY THEIR CONNATE PARAMETERS, RESULTING FROM THE SELECTED ADDITIVE COMBINATIONS OF THE REAL AND IMAGINARY COMPONENTS OF THE SIGNAL SPECTRAL DENSITY FUNCTION

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The investigation of sampling properties of a signal sample spectral density involves the statistical distributions of the real and imaginary components of the sampled signal. The distributions involved include the normal (Gaussian), the chi-square, the half normal, the truncated normal, and the Rayleigh distributions. We first review some of the properties of these distributions before and after (1) the absolute values of the real and imaginary components are summed, (2) when the absolute value of the sum of these components is taken, (3) when the squares of these components are summed, and (4) the square root of the sums of the squares is determined. These four distinct sums result in four different distributions. We compare of the respective means and variances of these distributions.

Simulations of sinusoidal signals added to noise with a standard normal distribution at five different sinusoidal amplitudes were evaluated for distribution type. The comparison shows that the distribution of the square root of the sums of the squares of the real and imaginary components has the smallest variance over those of the absolute value of the sums or the sum of the absolute values, though the statistical significance of the differences must be determined per sample size. When the difference is is not significant, the most convenient additive combination may be used.

Keywords: Complex analysis, signal analysis, spectral analysis, continuous distribution

1. INTRODUCTION

Weak signal acquisition, a long-time issue in radio astronomy, is now becoming more important in such applications as indoor global navigation satellite systems receivers. Increasing the sensitivity of the acquisition of weak signals is a critical problem. Receivers often use the squares of the real and imaginary components in the detection process, but the convenience of using the absolute values of the real and imaginary components suggests an analysis to evaluate if increased sensitivity (lower probability of false positive) for weak signal acquisition results. Hereafter, a weak signal is considered to be a sinusoidal signal with amplitude from one to five times the noise level - i.e., the signal to noise ratio (SNR) is from one to five. We leave for a future paper the case when the SNR is greater than zero but less than one. We evaluate the statistical properties of the squares and absolute values of the real and imaginary components including their relationships to the signal to noise ratio.

To investigate the sampling properties of the sample spectral density, we begin by denoting the real component as I^*, and denoting the imaginary component as Q^*. These components each have statistical distributions that, for closed form analytical purposes, follow the normal probability distribution function (pdf), though this is not necessary for the simulations described later. We evaluate the means and variances using the orthogonality properties that are known to hold for the type of spectra considered in this discussion. We use the property that I^* and Q^* each can have finite means and positive, nonzero variances. When I^* and Q^* are standardized, they have zero means and unit variances. These orthogonality properties lead to I^* and Q^* being uncorrelated and hence independent even as they are jointly bivariate normal. We also consider that each observation is jointly independent.

We know that the square of a standard normal distribution has a chi-square distribution with one degree of freedom, and that the sum of independent chi-square variables has a chi-square distribution with degrees of freedom added together (see the Appendix for details).

With these characteristics in mind, we first review some properties of the normal (Gaussian) distribution (Section 2). Then we focus on the investigation of the behaviors of I^* and Q^* when summed, when their absolute values are summed, and when the squares of I^* and Q^* are summed. These distinct sums result in a normal distribution, either a a half normal distribution or a truncated normal distribution (Section 3), and a Rayleigh distribution (Section 4). We then provide an elementary comparison of the respective means and variances of these distributions in Section 5. Finally, we present simulation studies of signals with noise against this incumbent noise in Section 6.

2. NORMAL DISTRIBUTION

Two random variables, I^* and Q^* have independently identically distributed (iid) normal distributions with parameters μ_{I^*}, $\sigma^2_{I^*}$, μ_{Q^*}, and $\sigma^2_{Q^*}$ if each has the respective pdfs

$$I^* \sim n\left(\mu_{I^*}, \sigma^2_{I^*}\right) = f_{I^*}\left(x^* | \mu_{I^*}, \sigma^2_{I^*}\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2_{I^*}}} e^{\left[-\left(x^* - \mu_{I^*}\right)^2/2\sigma^2_{I^*}\right]},$$
$$-\infty < x^* < \infty, \quad -\infty < \mu_{I^*} < \infty, \quad \sigma^2_{I^*} > 0,$$

and

$$Q^* \sim n\left(\mu_{Q^*}, \sigma^2_{Q^*}\right) = f_{Q^*}\left(y^* | \mu_{Q^*}, \sigma^2_{Q^*}\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2_{Q^*}}} e^{\left[-\left(y^* - \mu_{Q^*}\right)^2/2\sigma^2_{Q^*}\right]},$$
$$-\infty < y^* < \infty, \quad -\infty < \mu_{Q^*} < \infty, \quad \sigma^2_{Q^*} > 0,$$

where x^* and y^* are the realizations of the random variables I^* and Q^* respectively.

These distributions may be transformed into standard normal forms as follows:

$$x = \frac{x^* - \mu_{I^*}}{\sigma_{I^*}}, \text{ and } y = \frac{y^* - \mu_{Q^*}}{\sigma_{Q^*}};$$

which gives

$$I \sim n(0, 1) = f_I(x) = \frac{1}{\sqrt{2\pi}} e^{\left(-x^2/2\right)}, \quad -\infty < x < \infty,$$

and

$$Q \sim n(0, 1) = f_Q(y) = \frac{1}{\sqrt{2\pi}} e^{\left(-y^2/2\right)}, \quad -\infty < y < \infty,$$

where we let x and y be the realizations of the standard normal random variables I and Q, respectively. Hence, we may use the standard normal distribution to determine the distributions of the sums of the absolute values I and Q, and the sums of the squares of I and Q.

3. HALF NORMAL DISTRIBUTION

It can be shown that the sum of two normal distributions is normal with a mean that is the sum of each of the separate means, and variance that is the sum of each of the separate variances (Appendix). Thus,

$$I^* + Q^* \sim n\left(\mu_{I^*} + \mu_{Q^*}, \sigma^2_{I^*} + \sigma^2_{Q^*}\right).$$

If we use the standard normal form - viz. I and Q - we have, letting $W = I + Q$,

$$W \sim n\left(0, \sigma^2_I + \sigma^2_Q\right).$$

Equal variances give

$$W \sim n\left(0, 2\sigma^2\right), \quad \sigma^2 = \sigma^2_I = \sigma^2_Q,$$

and, of course, when W is from two standard normal distributions,

$$W \sim n(0, 2), \quad \sigma^2 = 1.$$

Let $V = |W|$. This results in "folding" the standard normal distribution negative values onto the positive values about zero. The pdf is

$$f_V\left(v | \sigma^2 = 2\right) = \frac{2}{\sqrt{2\pi\sigma^2}} e^{\left(-v^2/2\cdot\sigma^2\right)} = \frac{2}{\sqrt{2\pi 2}} e^{\left(-v^2/2\cdot 2\right)}$$
$$= \frac{1}{\sqrt{2}} \sqrt{\frac{2}{\pi}} e^{-v^{*2}/2}, \quad v^{*2} \geq 0, \quad v^{*2} = v^2/2.$$

So $f_V\left(v | \sigma^2\right)$ is a scaled half normal distribution with expected value (denoted *EV*, see Feller, 1966a,b, and the appendix)

$$EV = \frac{2}{\sqrt{2}} \sqrt{\frac{2}{\pi}} \sigma^2 = 4\sqrt{\frac{1}{\pi}} = 2.2567, \quad \sigma^2 = 2,$$

and variance (denoted *VarV*, see the appendix)

$$VarV = \frac{4}{2}\left(1 - \sqrt{\frac{2}{\pi}}\right)\sigma^2 = 4 - 4\sqrt{\frac{2}{\pi}} = 0.8085, \quad \sigma^2 = 2.$$

Now, suppose we use the standard normal forms of I and Q, take the absolute values of each, say I^{abs} and Q^{abs}, then sum these absolute values. We have, for $A = I^{abs} + Q^{abs} = |I| + |Q|$,

$$f_A\left(a | \sigma^2 = 1\right) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} e^{\frac{-x^2}{2\sigma^2}} + \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} e^{\frac{-y^2}{2\sigma^2}}$$
$$= \sqrt{\frac{2}{\pi}} e^{-x^2/2} + \sqrt{\frac{2}{\pi}} e^{-y^2/2}$$
$$a = |x| + |y| \geq 0, \quad \sigma^2 = 1.$$

So $f_A\left(a | \sigma^2\right)$ is the sum of two half normal distributions giving a one-sided truncated normal distribution with expected value (denoted *EA*)

$$EA = \sigma\sqrt{\frac{2}{\pi}} + \sigma\sqrt{\frac{2}{\pi}} = 2\sqrt{\frac{2}{\pi}} = 1.5958, \quad \sigma^2 = 1,$$

and variance (denoted *VarA*)

$$VarA=\sigma^2\left(1-\frac{2}{\pi}\right)+\sigma^2\left(1-\frac{2}{\pi}\right)=2\left(1-\frac{2}{\pi}\right)=0.7268,\ \ \sigma^2=1.$$

So, summing the two gaussian distributions resulting in a half normal distribution, then taking the absolute value results in larger expected values and variances than does taking the sum of the absolute values, each forming a half normal distribution, but when added results in a one-sided truncated normal distribution. The expected value of *A* is 29.29% smaller than the expected value of *V*, and the variance of *A* is 10.11% smaller than the variance of *V*.

4. RAYLEIGH DISTRIBUTION

We now turn to summing the squares of I and Q. Using the moment generating function (Appendix), it is straightforward to show that

$$I\sim\chi^2(1) \text{ and } Q\sim\chi^2(1),$$

which are the quadratic forms of I and Q and each have chi-square distributions with one degree of freedom (Appendix). It is well known (Appendix) that the sum of chi-square distributions is a chi-square distribution with the sum of each degrees of freedom summed. So, for $R^2=I^2+Q^2$,

$$R^2\sim\chi^2(2).$$

Therefore, for $\Gamma(\cdot)$ the gamma function,

$$f_{R^2}(r^2\,|\,p)=\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}\left(r^2\right)^{\frac{p}{2}-1}e^{-r^2/2},\ \ r>0,\ \ p=1,2,3,\ldots$$

For degrees of freedom *p*=2, we have,

$$f_{R^2}(r^2\,|\,p=2)=\frac{1}{\Gamma(1)2^1}\left(r^2\right)^0e^{-r^2/2}=\frac{1}{2}e^{-r^2/2}.$$

Now, $G(r)=P(\sqrt{R^2}\le\sqrt{r^2})=P(R\le r)$ is the cumulative distribution function (CDF) of $\sqrt{R^2}$. As R^2 is only positive or zero, $\sqrt{R^2}$ is also only positive or zero. Thus,

$$G(r)=P(R\le r)=\begin{Bmatrix}P(\sqrt{R^2}\le r), & r\ge 0\\ 0 &, \ \ else\end{Bmatrix}=\begin{Bmatrix}P(R^2\le r^2), & r\ge 0\\ 0 &, \ \ else\end{Bmatrix}.$$

For $r\ge 0$, $P(R^2\le r^2)=F_{R^2}(r^2)$, which is the CDF of R^2 evaluated at r^2. We now determine the CDF of R^2.

First, let $U=R^2$. Then,

$$f\left(R^2\,|\,p=2\right)=f\left(U\,|\,p=2\right)=\frac{1}{2}e^{-u/2},\ \ u\ge 0\,.$$

So

$$G(u)=\int_0^t f_U(u)\,du=1-e^{-t/2},\ \ t\ge 0\,.$$

To find $P(R^2\le r^2)$, we have that

$$G(u)=P(U\le u)=\begin{Bmatrix}1-e^{-u/2}, & u\ge 0\\ 0 & , \text{ else}\end{Bmatrix},$$

and therefore,

$$P(U\le u)=P(R^2\le r^2)=\begin{Bmatrix}1-e^{-r^2/2}, & r\ge 0\\ 0 & , \text{ else}\end{Bmatrix}.$$

The pdf of *R* is found by differentiating *G(u)* with respect to *u*. Thus,

$$g(u)=\frac{d}{du}G(u)=\begin{Bmatrix}\sqrt{u}\,e^{-u/2}, & u\ge 0\\ 0 & , \text{ else}\end{Bmatrix},$$

which results in

$$g(u)=g(r)=\begin{Bmatrix}r\,e^{-r^2/2}, & r\ge 0\\ 0 & , \text{ else}\end{Bmatrix},$$

and gives $g(r)\sim\text{Rayleigh}(\sigma^2=1)$. The mean of this distribution is

$$ER=\sqrt{\frac{\pi}{2}}\sigma^2=\sqrt{\frac{\pi}{2}}=1.2533,\ \ \sigma^2=1.$$

The variance the Rayleigh distribution is

$$VarR=\frac{4-\pi}{2}\sigma^2=\frac{4-\pi}{2}=0.4292,\ \ \sigma^2=1.$$

5. PARAMETER COMPARISON

The half normal, truncated normal, and the Rayleigh distributions resulting from combining I and Q have been derived and the connate parameters - the mean and variance - of each have been calculated for the case when the variance of I and Q is one. It remains to determine the "best" distribution to use with I and Q data.

As was shown above, the mean for the half normal distribution is $\mu_V=2.2567$, for the truncated normal distribution is $\mu_A=1.5958$, and the mean for the

Rayleigh distribution is $\mu_R = 1.2533$, when the variance for both I and Q individually is one. The Rayleigh distribution mean is the smallest. It should be noted that all three of the distributions have means that are linear functions of the their respective variances. That is,

$$V \sim half\ normal\left(\beta_V \sigma_V, \sigma_V^2\right),$$
$$A \sim truncated\ normal\left(\beta_A \sigma_A, \sigma_A^2\right),$$

and

$$R \sim Rayleigh\left(\beta_R \sigma_R, \sigma_R^2\right).$$

This is to say that, as the variance increases, so the mean also increases. This is important in it must be proven that the use of a likelihood ratio test procedure leads not only to a test statistic, but also to maximum likelihood estimates for the parameters for the unequal sample size case when comparing two or more samples. This proof, however, is beyond the scope of this paper.

The values of the variance of the half normal distribution is $\sigma_V^2 = 0.8085$, the truncated normal variance is $\sigma_A^2 = 0.7268$, and the value of the variance of the Rayleigh distribution is $\sigma_R^2 = 0.4292$, when the variances of I and Q individually are one. The Rayleigh distribution has the smallest variance. This is particularly important as tests of hypotheses, which frequently need the variance to evaluate, will result in greater sensitivity if the Rayleigh distribution, and hence its variance, is used.

Table 1: Comparison of the combinations of I and Q: $V = |I + Q|$ *to obtain a half normal distribution,* $A = |I| + |Q|$ *for a truncated normal distribution, and* $R = \sqrt{I^2 + Q^2}$ *for a Rayleigh distribution.*

Distribution	Mean	Variance
V	2.2567	0.8085
A	1.5958	0.7268
R	1.2533	0.4292

6. SIMULATION

We now simulate the real and imaginary signal compnents

$$I = S * \sin(\Phi) + n(0,1)$$
$$Q = S * \cos(\Phi) + n(0,1)$$

where S is a constant that successively takes the values 0, 1, 2, 5 and 10, Φ is a uniform random variable such that $\Phi \sim unif(-\pi, \pi)$ and the noise term is a standard normal distribution. Note when S = 0, we are simulating noise with no signal. Samples totaling 500 for each of the two normal distributions, and the sine and cosine terms were drawn. The respective components then were added to form I and Q as in the equations above. Illustrations 1 through 5 show the real component I with S = 0, 1, 2, 5, and then 10. The imaginary component Q has similar distributions so it is not shown. The illustrations show that the noise for each value of *S* remains $n(0,1)$. The sinusoidal signal has the same basic distribution for all *S*, though the density is proportionally smaller as *S* increases. This is due to normalizing the signal amplitude to one. In the case of *S* = 0, the signal is constant at zero and hence a uniform distribution is portrayed. We therefore can ignore the kernel density fit line.

Now, with the sum of the sinusoidal signal and the normal noise, we see that, as the signal becomes dominant, the distribution reshapes from a normal distribution, to a distribution that is distinctly bimodal; i.e., two peaks. So for sinusoidal signals that have amplitudes between one and five times the noise level, we can estimate the signal plus noise with a normal distribution. This allows us to utilize the results in Sections 2 though 5 above. For sinusoidal signals that have amplitudes greater than five, we have a non-normal distribution, and closed form analytics are much more difficult than what we have already developed. However, if we restrict ourselves to weak signals - amplitudes of 1 to 5 - this difficulty is circumvented.

Finally, I and Q were combined in the three ways described in the earlier Sections 3 and 4. Illustrations 6 through 10 show the distributions of the the three combinations of I and Q including the distribution of $R^2 = I^2 + Q^2$, for each of the five levels of the amplitude term *S*. Table 2 shows that for each level of *S*, *R* has the smallest variance of the combination methods. Again for each level of *S*, the smallest variance is for the square root of the sums of the squares (*R*) of *I* and *Q*. This is reinforced by the ranges (max - min). The variance of the absolute value of the sum (*V*) of I and Q is smaller than the sum of the absolute values (*A*), though a test of the variances will show that the ratios of these variances may be statistically significant based upon sample size. Note that the standard *F*-test is not necessarily appropriate as the distributions of the combinations necessarily do not follow a normal pdf.

For the cases *S* = 0, 1, and 2, a visual inspection Illustrations 6, 7, and 8 indicate that the distributions for *A*, *V*, *R2*, and *R* follow the closed form distribution analysis in Sections 2, 3, 4; i.e., *A* is a truncated normal distribution, *V* is a half normal distribution, *R2* is a chi-square distribution, and *R* is a Rayleigh distribution. For the cases of *S* = 5 and 10, we see that possibly significant departures from the closed form analysis results are apparent. Although we are not especially interested in these two cases, it is of future interest to investigate the use of the three-parameter Weibull distribution to fit not only these two cases for all the sums, but also the three other cases of *S* for all the sums. Although the Weibull distribution may estimate all the sums at all the levels of S, we anticipate the three parameters that shape the

Weibull distribution will be different for each case.

Table 2: Sums statistics of the combinations of I and Q: $V=|I+Q|$, $A=|I|+|Q|$, $R2=I^2+Q^2$, *and* $R=\sqrt{I^2+Q^2}$. *S is the amplitude of the sinusoidal signal, and N is the size of the sample.*

S	Sum	Median	Mean	Var	Min	Max	N
0	A	1.5073	1.6089	0.7261	0.0391	5.0347	500
0	V	0.9215	1.1099	0.7564	0.0096	4.3996	500
0	R2	1.3862	1.9960	3.8796	0.0014	12.9934	500
0	R	1.1774	1.2589	0.4119	0.0377	3.6046	500
1	A	1.5197	1.6210	0.6826	0.0611	4.7099	500
1	V	1.0141	1.1559	0.7135	0.0010	4.7099	500
1	R2	1.4475	1.9960	3.3111	0.0029	11.1552	500
1	R	1.2031	1.2673	0.3908	0.0535	3.3399	500
2	A	1.5612	1.6540	0.6324	0.1006	4.1778	500
2	V	0.9881	1.1278	0.6401	0.0007	4.1778	500
2	R2	1.5121	1.9960	2.9060	0.0051	9.1559	500
2	R	1.2297	1.2821	0.3528	0.0711	3.0259	500
5	A	1.7847	1.7619	0.4539	0.1245	3.3423	500
5	V	1.0197	1.1556	0.6500	0.0022	3.3423	500
5	R2	1.8353	1.9960	1.4449	0.0120	5.5855	500
5	R	1.3547	1.3355	0.2128	0.1097	2.3634	500
10	A	1.7975	1.7877	0.3772	0.1770	3.1180	500
10	V	0.9684	1.0909	0.6040	0.0010	3.0631	500
10	R2	1.9760	1.9960	1.0833	0.0257	4.9410	500
10	R	1.4057	1.3533	0.1650	0.1602	2.2228	500

7. CONCLUSION

To investigate the sampling properties of the sample spectral density, we began by noting that the real component I^*, and the imaginary component Q^*, each can have statistical distributions that follow the normal probability distribution function, which was assumed for the closed form analysis, and not necessarily so for the simulations. We then reviewed some properties of the normal distribution, the half normal distribution, the truncated normal distribution, and the Rayleigh distribution. Then we focused on the investigation of the behaviors of I and Q (the standardized versions of I^* and Q^*) when summed, and when the squares of I and Q are summed. These sums resulted in four distinct sums with a half normal distribution, a truncated distribution, a chi-square distribution, and a Rayleigh distribution. (It was noted that a three-parameter Weibull distribution might be used to estimate all four of these distributions.) We concluded with an elementary comparison of the means and variances of these distributions.

The comparison showed that the square root of the sums of the squares (*R*) of I and Q has smaller variance than the absolute value of the sum (*V*) of I and Q and the sum of the absolute values (*A*) of I and Q, leading us to conclude that the desirable distribution is R for testing hypotheses of means and variances. This smaller variance leads to increased test sensitivity (a signal is detected when a signal is present with noise) and selectivity (no signal is detected when only noise is present), thereby giving smaller probabilities of false positive errors and false negative errors. Also, it was shown that the variance of *A* is smaller than the variance of *V*, though the significance of the difference varies with sample size and the level of *S*. If the difference of the variances of R and A are not significant for a specified sample size, then either the square root of the sums of the squares or the sum of the absolute values of the real and imaginary components may used. Computation overhead is the most likely benefit for using one sum combination over another.

APPENDIX

For details of the definitions and theorems below, see Mood, Graybill, and Boes (1966), or any calculus-based text on mathematical statistics.

Definition 1. Expected value. The expected value or mean of a random variable, say *g(X)*, denoted by *Eg(X)*, is, for pdf $f_X(x)$,

$$E\,g(X)=\int_{-\infty}^{\infty} g(x)f(x)\,dx,\quad X \text{ continuous}$$

provided the integral exists. If $E|g(X)|=\infty$, we say the expected value does not exist.

Definition 2. Variance. The variance of a random variable *g(X)* is $Var(g(X))=EX^2-(EX)^2$. The positive square root of the variance is the standard deviation of *g(X)*.

Definition 3. Moment generating function (mgf). Let X be a random variable with pdf $f_X(\cdot)$. The expected value of e^{tX} is defined to be the moment generating function of X if the expected value exists for every value of t in some interval $-h<t<h,\ h>0$. The moment generating function, denoted by $m_X(t)$ or $m(t)$, is

$$m(t)=E(e^{tX})=\int_{-\infty}^{\infty} f(x)\,dx$$

if the random variable X is continuous. Note that the mgf is the Laplace transform of $f_X(\cdot)$.

Theorem 1. If $X_1,\ldots,X_n$ are independent random variables and the moment generating function of each exists for all $-h<t<h,\ h>0$, let $Y=\sum_1^n X_i$; then

$$m_Y(t)=E\left(e^{\sum tX_i}\right)=\prod_{i=1}^{n} m_{X_i}(t),\ \text{for } -h<t<h.$$

Proof:

$$m_Y(t) = E\left(e^{\sum tX_i}\right) = E\left[\prod_{i=1}^{n} m_{X_i}(t)\right] = \prod_{i=1}^{n} E\left(e^{tX_i}\right) = \prod_{i=1}^{n} m_{X_i}(t).$$

This theorem states that the product of the moment generating function of a series of distributions results in a distribution that is the sum of the random variables. The following lemma is a statement that any linear combinations of independent normal random variables is a normally distributed random variable.

Lemma. Assume that $X_1, \ldots, X_n$ are independent random variables and $X_i \sim n(\mu_i,\ \sigma_i^2);$ then the sum of iid normal random variables is a normal random variable.

Proof: From $X_i \sim n(\mu_i,\ \sigma_i^2);$

$$a_i X_i \sim n(a_i \mu_i,\ a_i^2 \sigma_i^2),$$

and

$$m_{a_i X_i}(t) = \exp\left(a_i \mu_i t + \frac{1}{2} a_i^2 \sigma_i^2 t^2\right).$$

Hence

$$m_{\sum a_i X_i}(t) = \prod_{i=1}^{n} m_{a_i X_i}(t) = \exp\left[\left(\sum a_i \mu_i\right) t + \frac{1}{2}\left(a_i^2 \sigma_i^2\right) t^2\right],$$

which is the moment generating function of a normal random variable and distribution is

$$\sum_{1}^{n} a_i X_i \sim n\left(\sum_{1}^{n} a_i \mu_i,\ \sum_{1}^{n} a_i^2 \sigma_i^2\right).$$

The following theorem says that the sum of the squares of independent standard normal random variables has a chi-square distribution with degrees of freedom equal to the number of terms in the sum.

Theorem 2. If the random variables $X_1, \ldots, X_k,$ are normally and independently distributed with means μ_i and variances $\sigma_i^2,$ then

$$U = \sum_{i=1}^{k} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$$

has a chi-square distribution with k degrees of freedom.

Proof: When $Z_i = X_i - \mu_i / \sigma_i,$ Z_i has a standard normal distribution. As

$$m_U(t) = E\left(e^{tU}\right) = E\left[e^{t \sum Z_i^2}\right] = E\left[\prod_{i=1}^{n} e^{tZ_i^2}\right] = \prod_{i=1}^{n} E\left[e^{tZ_i^2}\right].$$

Now

$$\begin{aligned} E\left(e^{tZ_i^2}\right) &= \int_{-\infty}^{\infty} e^{tz^2}\left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}z^2}\, dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1-2t)z^2}\, dz \\ &= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{\sqrt{1-2t}}{\sqrt{2\pi}} e^{-\frac{1}{2}(1-st)z^2}\, dz \\ &= \frac{1}{\sqrt{1-2t}},\ \text{for } t < \frac{1}{2}, \end{aligned}$$

the last integral is the area under a normal distribution with variance $1/(1-2t)$ and evaluates to unity. Therefore,

$$\prod_{i=1}^{k} E\left(e^{tZ_i^2}\right) = \prod_{i=1}^{k} \frac{1}{\sqrt{1-2t}} = \left(\frac{1}{\sqrt{1-2t}}\right)^{\frac{k}{2}} \text{ for } t < \frac{1}{2},$$

which is the mgf of a chi-square distribution with *k* degrees of freedom.

We can use Theorem 1 so show that the sum of independently distributed chi-square random variables, each with $k_i,\ i = 1, 2, \ldots, n$ degrees of freedom, is a chi-square distribution with $k = \sum_{i=1}^{i} (k_i)$ degrees of freedom.

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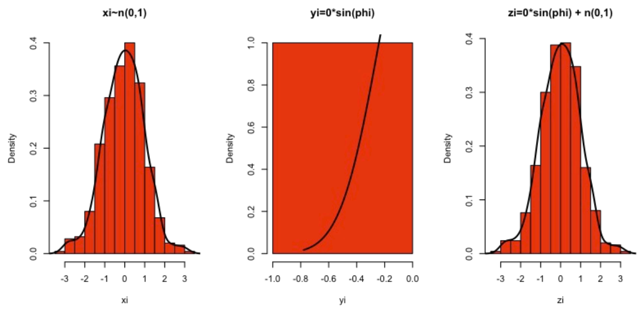

Illustration 1: Combinations of I for S = 0, n = 500. The distributions for Q are similar.

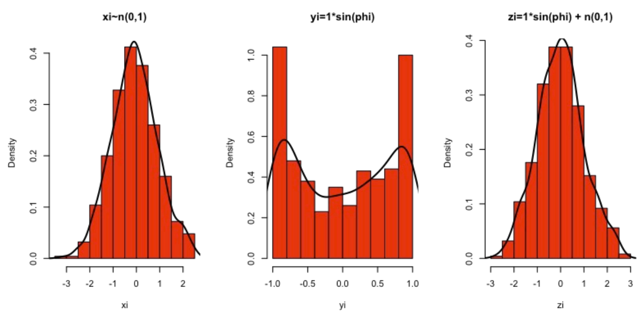

Illustration 2: Combinations of I for S = 1, n = 500. The distributions for Q are similar.

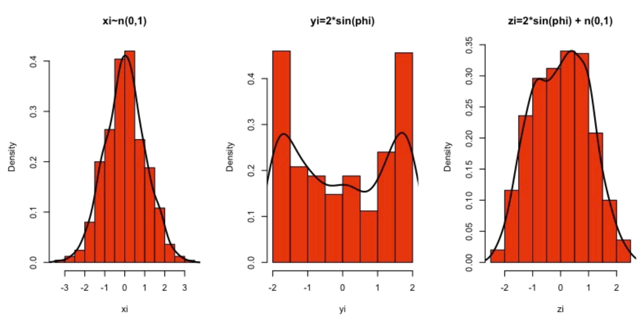

Illustration 3: Combinations of I for S = 2, n = 500. The distributions for Q are similar.

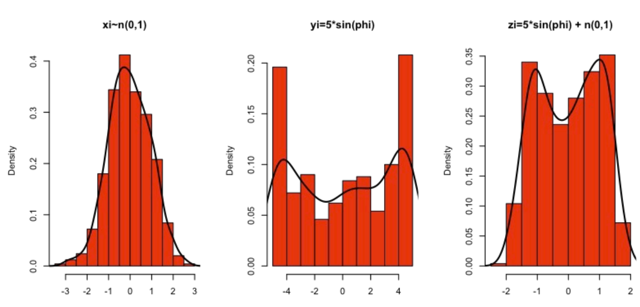

Illustration 4: Combinations of I for S = 5, n = 500. The distributions for Q are similar.

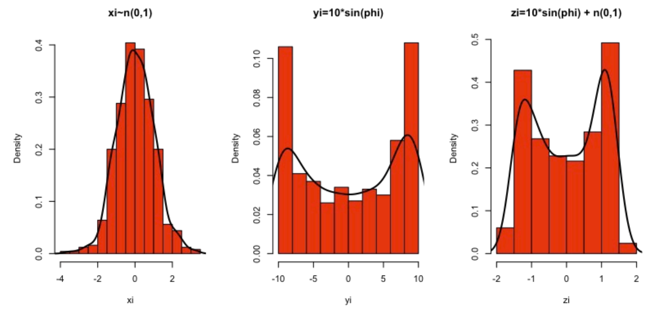

Illustration 5: Combinations of I for S = 10, n = 500. The distributions for Q are similar.

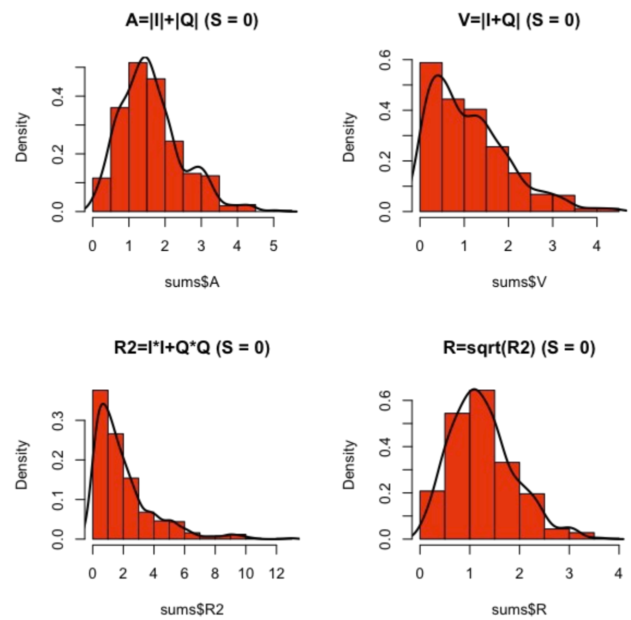

Illustration 6: Combinations of I and Q for S = 0, n = 500. The Rayleigh distribution R has the smallest variance.

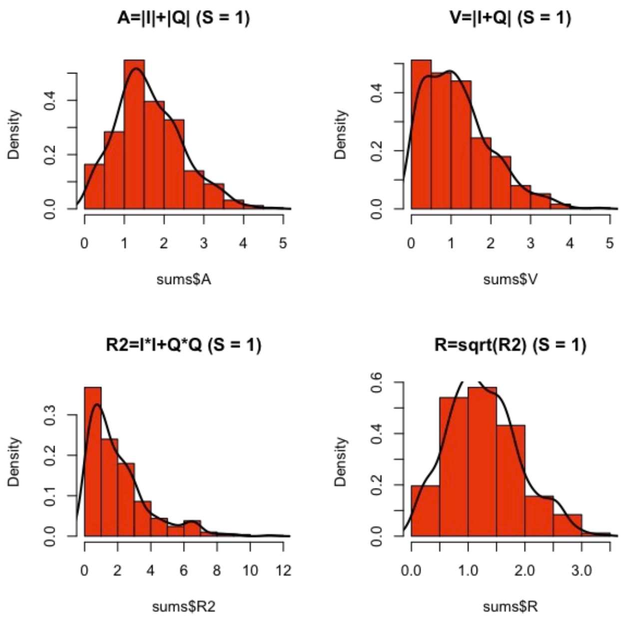

Illustration 7: Combinations of I and Q for S = 1, n = 500. The Rayleigh distribution R has the smallest variance.

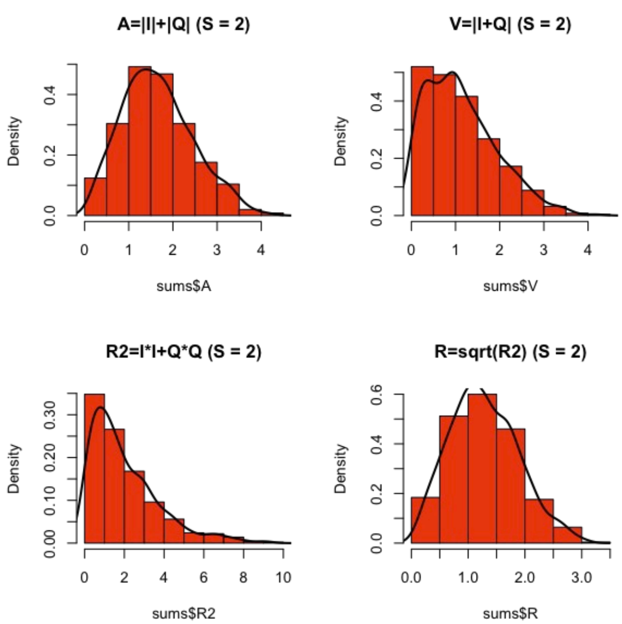

Illustration 8: Combinations of I and Q for S = 2, n = 500. The Rayleigh distribution R has the smallest variance.

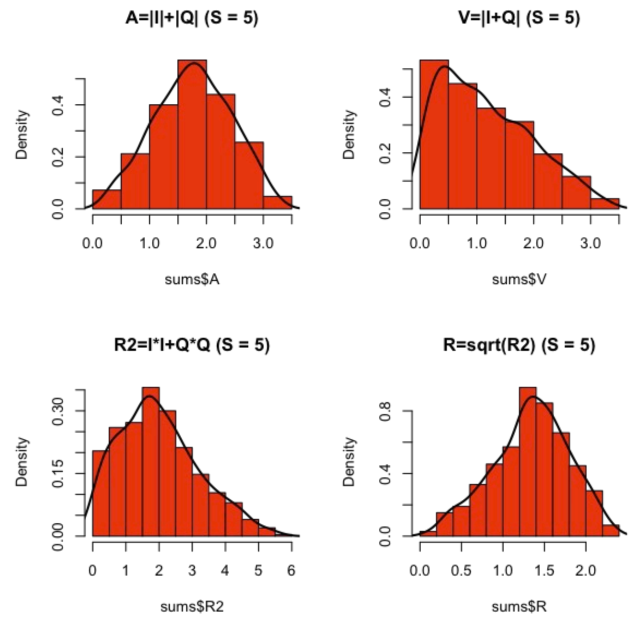

Illustration 9: Combinations of I and Q for S = 5, n = 500. The Rayleigh distribution R has the smallest variance.

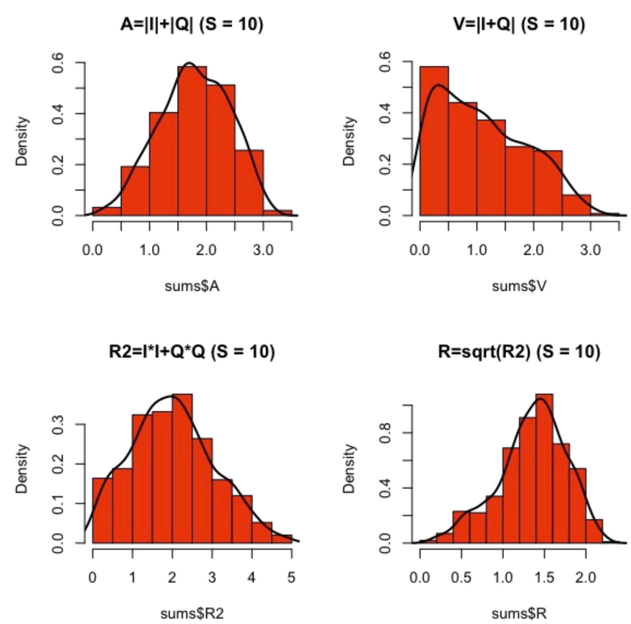

Illustration 10: Combinations of I and Q for S = 10, n = 500. The Rayleigh distribution R has the smallest variance.